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Probability Theory and Applications (MA208)
Problem Sheet - 4

One-Dimensional Random Variables

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1. A coin is known to come up heads three times as often as tails. This coin is tossed three times. Let X be the number of heads that appear. Write out the probability distribution of X and also the cdf. Make a sketch of both.

2. From a lot containing 25 items, 5 of which are defective, 4 are chosen at random. Let X be the number of defectives found. Obtain the probability distribution of X if
   (a) the items are chosen with replacement,
   (b) the items are chosen without replacement.

3. Suppose that the random variable X has possible values $1, 2, 3, \ldots$, and $P(X = j) = 1/2^j$, $j = 1, 2, \ldots$
   (a) Compute $P(X \text{ is even})$.
   (b) Compute $P(X \geq 5)$.
   (c) Compute $P(X \text{ is divisible by } 3)$.

4. Consider a random variable X with possible outcomes: $0, 1, 2, \ldots$. Suppose that $P(X = j) = (1 - a)a^j$, $j = 0, 1, 2, \ldots$
   (a) For what values of a is the above model meaningful?
   (b) Verify that the above does represent a legitimate probability distribution.
   (c) Show that for any two positive integers s and t,
   $$P(X > s + t | X > s) = P(X \geq t).$$

5. Suppose that twice as many items are produced (per day) by machine 1 as by machine 2. However, about 4 percent of the items from machine 1 tend to be defective while machine 2 produces only about 2 percent defectives. Suppose that the daily output of the two machines is combined. A random sample of 10 is taken from the combined output. What is the probability that this sample contains 2 defectives?

6. Rockets are launched until the first successful launching has taken place. If this does not occur within 5 attempts, the experiment is halted and the equipment inspected. Suppose that there is a constant probability of 0.8 of having a successful launching and that successive attempts are independent. Assume that the cost of the first launching is K dollars while subsequent launchings cost $K/3$ dollars. Whenever a successful launching takes place, a certain amount of information is obtained which may be expressed as financial gain of, say C dollars. If T is the net cost of this experiment, find the probability distribution of T.

7. Evaluate $P(X = 5)$, where X is the random variable defined in Example 4.10. Suppose that $n_1 = 10$, $n_2 = 15$, $p_1 = 0.3$ and $p_2 = 0.2$.

8. (*Properties of the binomial probabilities.*) In the discussion of Example 4.8 a general pattern for the binomial probabilities $\binom{n}{k}p^k(1-p)^{n-k}$ was suggested. Let us denote these probabilities by $p_n(k)$.

   (a) Show that for $0 \le k < n$ we have

   $$p_n(k+1)/p_n(k) = [(n-k)/(k+1)][p/(1-p)].$$

   (b) Using (a) show that

      i. $p_n(k+1) > p_n(k)$ if $k < np - (1-p)$,
      ii. $p_n(k+1) = p_n(k)$ if $k = np - (1-p)$,
      iii. $p_n(k+1) < p_n(k)$ if $k > np - (1-p)$.

   (c) Show that if $np - (1-p)$ is an integer, $p_n(k)$ assumes its maximum value for two values of k, namely $k_0 = np - (1-p)$ and $k_0' = np - (1-p) + 1$.

   (d) Show that if $np - (1-p)$ is not an integer then $p_n(k)$ assumes its maximum value when k is equal to the smallest integer greater than k_0.

   (e) Show that if $np - (1-p) < 0$, $p_n(0) > p_n(1) > \cdots > p_n(n)$ while if $np - (1-p) = 0$, $p_n(0) = p_n(1) > p_n(2) > \cdots > p_n(n)$.

9. The continuous random variable X has pdf $f(x) = x/2, 0 \le x \le 2$. Two independent determinations of X are made. What is the probability that both these determinations will be greater than one? If three independent determinations had been made, what is the probability that exactly two of these are larger than one?

10. Let X be the life length of an electron tube and suppose that X may be represented as a continuous random variable with pdf $f(x) = be^{-bx}$, $x \ge 0$. Let $p_j = P(j \le X < j+1)$. Show that p_j is of the form $(1-a)a^j$ and determine a.

11. The continuous random variable X has pdf $f(x) = 3x^2$, $-1 \le x \le 0$. If b is a number satisfying $-1 < b < 0$, compute $P(X > b | X < b/2)$.

12. Suppose that f and g are pdf's on the same interval, say $a \le x \le b$.

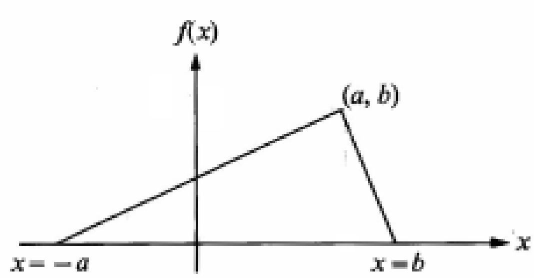
   (a) Show that $f + g$ is *not* a pdf on that interval.

   (b) Show that for every number β, $0 < \beta < 1$, $\beta f(x) + (1-\beta)g(x)$ is a pdf on that interval.

13. Suppose that the graph in Fig. 4.16 represents the pdf of a random variable X.

   (a) What is the relationship between a and b?

   (b) If $a > 0$ and $b > 0$, what can you say about the largest value which b may assume? (See the following figure.)



14. The percentage of alcohol $(100X)$ in a certain compound may be considered as a random variable, where $X, 0 < X < 1$, has the following pdf:

$$f(x) = 20x^3(1-x), \quad 0 < x < 1.$$

(a) Obtain an expression for the cdf F and sketch its graph.

(b) Evaluate $P(X \leq \frac{2}{3})$.

(c) Suppose that the selling price of the above compound depends on the alcohol content. Specifically, if $\frac{1}{3} < X < \frac{2}{3}$, the compound sells for C_1 dollars/gallon; otherwise it sells for C_2 dollars/gallon. If the cost is C_3 dollars/gallon, find the probability distribution of the net profit per gallon.

15. Let X be a continuous random variable with pdf f given by:

$$\begin{aligned} f(x) &= ax, && 0 \leq x \leq 1, \\ &= a, && 1 \leq x \leq 2, \\ &= -ax + 3a, && 2 \leq x \leq 3, \\ &= 0, && \text{elsewhere.} \end{aligned}$$

(a) Determine the constant a.

(b) Determine F, the cdf, and sketch its graph.

(c) If X_1, X_2, and X_3 are three independent observations from X, what is the probability that exactly one of these three numbers is larger than 1.5?

16. The diameter on an electric cable, say X, is assumed to be a continuous random variable with pdf $f(x) = 6x(1-x), 0 \leq x \leq 1$.

(a) Check that the above is a pelf and sketch it.

(b) Obtain an expression for the cdf of X and sketch it.

(c) Determine a number b such that $P(X < b) = 2P(X > b)$.

(d) Compute $P(X \leq \frac{1}{2} | \frac{1}{3} < X < \frac{2}{3})$.

17. Each of the following functions represents the cdf of a continuous random variable. In each case $F(x) = 0$ for $x < a$ and $F(x) = 1$ for $x > b$, where $[a, b]$ is the indicated interval. In each case, sketch the function F, determine the pdf f and sketch it. Also verify that f is a pdf.

(a) $F(x) = x/5, 0 \leq x \leq 5$

(b) $F(x) = (2/\pi) \sin^{-1}(\sqrt{x}), 0 \leq x \leq 1$

(c) $F(x) = e^{3x}, -\infty < x \leq 0$

(d) $F(x) = x^3/2 + \frac{1}{2}, -1 \leq x \leq 1$.

18. Let X be the life length of an electronic device (measured in hours). Suppose that X is a continuous random variable with pdf $f(x) = k/x^n$, $2000 \leq x \leq 10,000$.

(a) For $n = 2$, determine k.

(b) For $n = 3$, determine k.

(c) For general n, determine k.

(d) What is the probability that the device will fail before 5000 hours have elapsed?

(e) Sketch the cdf $F(t)$ for (c) and determine its algebraic form.

19. Let X be a binomially distributed random variable based on 10 repetitions of an experiment. If $p = 0.3$, evaluate the following probabilities using the table of the binomial distribution in the Appendix.

    (a) $P(X \leq 8)$
    (b) $P(X = 7)$
    (c) $P(X > 6)$.

20. Suppose that X is uniformly distributed over $[-\alpha, +\alpha]$, where $\alpha > 0$. Whenever possible, determine α so that the following are satisfied.

    (a) $P(X > 1) = \frac{1}{3}$
    (b) $P(X > 1) = \frac{1}{2}$
    (c) $P(X < \frac{1}{2}) = 0.7$
    (d) $P(X < \frac{1}{2}) = 0.3$
    (e) $P(|X| < 1) = P(|X| > 1)$.

21. Suppose that X is uniformly distributed over $[0, \alpha]$, $\alpha > 0$. Answer the questions of Problem 20.

22. A point is chosen at random on a line of length L. What is the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$?

23. A factory produces 10 glass containers daily. It may be assumed that there is a constant probability $p = 0.1$ of producing a defective container. Before these containers are stored they are inspected and the defective ones are set aside. Suppose that there is a constant probability $r = 0.1$ that a defective container is misclassified. Let X equal the number of containers classified as defective at the end of a production day. (Suppose that all containers which are manufactured on a particular day are also inspected on that day.)

    (a) Compute $P(X = 3)$ and $P(X > 3)$.
    (b) Obtain an expression for $P(X = k)$.

24. Suppose that 5 percent of all items coming off a production line are defective. If 10 such items are chosen and inspected, what is the probability that at most 2 defectives are found?

25. Suppose that the life length (in hours) of a certain radio tube is a continuous random variable X with pdf $f(x) = 100/x^2$, $x > 100$, and 0 elsewhere.

    (a) What is the probability that a tube will last less than 200 hours if it is known that the tube is still functioning after 150 hours of service?
    (b) What is the probability that if 3 such tubes are installed in a set, exactly one will have to be replaced after 150 hours of service?
    (c) What is the maximum number of tubes that may be inserted into a set so that there is a probability of 0.5 that after 150 hours of service all of them are still functioning?

26. An experiment consists of n independent trials. It may be supposed that because of "learning," the probability of obtaining a successful outcome increases with the number of trials performed. Specifically, suppose that $P(\text{success on the } i\text{th repetition}) = (i+1)/(i+2)$, $i = 1, 2, \ldots, n$.

(a) What is the probability of having at least 3 successful outcomes in 8 repetitions?

(b) What is the probability that the first successful outcome occurs on the eighth repetition?

27. Referring to Example 4.10,

    (a) evaluate $P(X = 2)$ if $n = 4$,

    (b) for arbitrary n, show that $P(X = n - 1) = P(\text{exactly one unsuccessful attempt})$ is equal to $[1/(n+1)] \sum_{i=1}^{n} (1/i)$.

28. If the random variable K is uniformly distributed over $(0, 5)$, what is the probability that the roots of the equation $4x^2 + 4xK + K + 2 = 0$ are real?

29. Suppose that the random variable X has possible values $1, 2, 3, \ldots$ and that $P(X = r) = k(1 - \beta)^{r-1}$, $0 < \beta < 1$.

    (a) Determine the constant k.

    (b) Find the *mode* of this distribution (i.e., that value of r which makes $P(X = r)$ largest).

30. A random variable X may assume four values with probabilities $(1 + 3x)/4$, $(1 - x)/4$, $(1 + 2x)/4$, and $(1 - 4x)/4$. For what values of x is this a probability distribution?

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